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Upper bounds to the zero-field susceptibilities and magnetisations for the random-bond Ising model and the random-bond n-vector model

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Abstract. The self-avoiding walk approximation is used to give upper bounds to the zero-field spin-pair correlation function, zero-field susceptibilities and the spontaneous magnetisation for the systems of the random-bond Ising model of general spin S and the random-bond n-vector model, where the exchange integrals are assumed to be random variables. Using the obtained bounds, we find a sufficient condition for disappearance of the long-range orders. It turns out that the condition is stronger for some cases than the one obtained in a previous paper by the present authors. The present one applies to some cases for which the preceding one does not.

1. Introduction

In a previous paper (Horiguchi and Morita 1981, to be referred to as I), we obtained upper bounds to the spin correlation functions in the thermodynamic limit of the zero external field limit for the random-bond Ising model of general spin S and for the random-bond n-vector model, in both of which the exchange integrals are quenched random variables and take on J ($J>0$), 0 and $-J$ with probabilities p, r and $1-p-r$, respectively. By using the upper bounds obtained to the spontaneous magnetisation (or spontaneous sublattice magnetisation), we found the lower (or upper) bound to the critical concentration of the ferromagnetic bonds for disappearance of the ferromagnetic (or antiferromagnetic) state. These results of paper I were extended to the systems with exchange integrals whose probability distributions are continuous. We then found a more general form of the sufficient condition for disappearance of the long-range orders (Horiguchi and Morita 1982, to be referred to as II). However, in paper II, we made a restriction to the probability distributions $P_{ij}(J_{ij})$ of the exchange integrals J_{ij}, that is, $P_{ij}(J_{ij})$ is zero whenever $P_{ij}(-J_{ij})$ is zero. Because of this restriction, we could not discuss the systems with the exchange integrals whose probability distribution is, for example, the rectangular distribution in the interval $[-a, b]$ except when $a=b$. The random-bond Ising model with such a distribution of J_{ij}, with $a \neq b$, was investigated as a model of the spin glass by Katsura (1977) in the Bethe approximation. The primary aim of this paper is to remove this restriction imposed in paper II in order to cover such a case.

Fisher and Sykes (1959) and Fisher (1967) showed that the self-avoiding walk approximation gives upper bounds to the zero-field spin-pair correlation functions, to

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the zero-field susceptibility and to the critical temperature for the ferromagnetic Ising model. For the random-bond and random-site ferromagnetic Ising model, Morita (1979) showed that these upper bounds are expressed in terms of the corresponding quantities for the regular Ising model. It was also noted that these give upper bounds for the random-bond Ising model in which the exchange integrals take on negative as well as positive values.

In the present paper, we use these facts in order to remove the restriction imposed in paper II on the probability distributions of J_{ij}. We consider the systems of the random-bond Ising model of general spin S and of the random-bond n-vector model. First, we obtain upper bounds to the zero-field spin-pair correlation functions and to the zero-field susceptibilities in § 2. We obtain upper bounds to the spontaneous long-range order parameters and a sufficient condition for disappearance of the long-range orders in § 3. In § 4, the condition is discussed for several types of the probability distribution. It turns out that the condition is stronger than the one in paper II for some cases, even when the latter applies. Concluding remarks are given in § 5.

2. Upper bounds to zero-field spin-pair correlation functions and to zero-field susceptibilities

First we consider the system of the random-bond Ising model of general spin S under zero external field on a finite set of N lattice sites

$$H_S^{(0)} = -\sum_{(ij)} J_{ij}s_is_j \tag{2.1}$$

where s_i takes on the values $-S, -S+1, \ldots, S$. J_{ij} are mutually independent, quenched random variables and their probability distributions are denoted by $\tilde{P}_{ij}(J_{ij})$. We are concerned with the configurational average of the spin-pair correlation functions $\langle\langle s_ks_l\rangle_{N,0}^{\{\beta J\}}\rangle_c$ for $k \neq l$, where

$$\langle s_ks_l\rangle_{N,0}^{\{\beta J\}} = \mathrm{Tr}\, s_ks_l \exp(-\beta H_S^{(0)})/\mathrm{Tr}\exp(-\beta H_S^{(0)}). \tag{2.2}$$

Here $\beta = 1/k_\mathrm{B}T$ as usual and k_B is the Boltzmann constant. The angular brackets with a suffix c denote the configurational average of a function of $\{J_{ij}\}$

$$\langle Q\{J_{ij}\}\rangle_c = \int \left(\prod_{(ij)} \mathrm{d}J_{ij}\right) P\{J_{ij}\}Q\{J_{ij}\} \tag{2.3}$$

where

$$P\{J_{ij}\} = \prod_{(ij)} \tilde{P}_{ij}(J_{ij}). \tag{2.4}$$

We denote the set of J_{ij} for which $\tilde{P}(J_{ij})+\tilde{P}(-J_{ij})$ is non-zero by $\mathscr{J}_{ij}$ and the set of J_{ij} for which $\tilde{P}(J_{ij})+\tilde{P}(-J_{ij})$ has a delta-function singularity by $\mathscr{J}_{ij}^*$. We take a function $\zeta(J_{ij})$ which is positive and integrable in the set $\mathscr{J}_{ij}$ and has a delta-function singularity at J_{ij} belonging to the set J_{ij}^*, and assume that it satisfies

$$\int \zeta(J_{ij})\,\mathrm{d}J_{ij} \equiv N_\zeta < \infty.$$

If $\mathscr{J}_{ij}$ is of finite measure and $\mathscr{J}_{ij}^*$ is a finite set, $\zeta(J_{ij})$ is chosen to be identically unity for $J_{ij} \in \mathscr{J}_{ij}$ and has a delta-function singularity with amplitude unity at J_{ij} belonging to the set $\mathcal{J}_{ij}^*$, and zero otherwise. For a small positive number ε, we introduce a function $\tilde{P}_{ij}^{(\varepsilon)}(J_{ij})$ by

$$\tilde{P}_{ij}^{(\varepsilon)}(J_{ij}) = \frac{1}{(1+\varepsilon N_\zeta)}[P_{ij}(J_{ij}) + \varepsilon\zeta(J_{ij})]. \tag{2.5}$$

We now define the quantities $\alpha_{ij}^{(\varepsilon)}$ and $\beta_{ij}^{(\varepsilon)}$ by

$$\alpha_{ij}^{(\varepsilon)} \equiv \alpha_{ij}^{(\varepsilon)}(|J_{ij}|) = [\tilde{P}_{ij}^{(\varepsilon)}(J_{ij})\tilde{P}_{ij}^{(\varepsilon)}(-J_{ij})]^{1/2}$$

$$\beta_{ij}^{(\varepsilon)} \equiv \beta_{ij}^{(\varepsilon)}(|J_{ij}|) = \begin{cases} 0 & J_{ij} = 0 \text{ or } J_{ij} \notin \mathcal{J}_{ij} \cup \mathcal{J}_{ij}^* \\ (1/2J_{ij}) \ln[\tilde{P}_{ij}^{(\varepsilon)}(J_{ij})/\tilde{P}_{ij}^{(\varepsilon)}(-J_{ij})] & \text{otherwise.} \end{cases} \tag{2.7}$$

We agree that the limiting procedure tending ε to zero is taken before the thermodynamic limit.

Introducing dichotomic variables $\{\sigma_i\}$ each of which takes ± 1 and performing the gauge transformation (Horiguchi 1981), we obtain

$$\langle\langle s_k s_l\rangle_{N,0}^{\{\beta J\}}\rangle_c = \lim_{\varepsilon\to 0} \int \left(\prod_{(ij)} \mathrm{d}J_{ij}\right) P\{J_{ij}\}\langle\sigma_k\sigma_l\rangle_{N,0}^{\{\beta^{(\varepsilon)}J\}}\langle s_k s_l\rangle_{N,0}^{\{\beta J\}} \tag{2.8}$$

where the superscripts $\{\beta J\}$ and $\{\beta^{(\varepsilon)}J\}$ represent the sets of $\{\beta J_{ij}\}$ and $\{\beta_{ij}^{(\varepsilon)}J_{ij}\}$, respectively, and

$$\langle\sigma_k\sigma_l\rangle_{N,0}^{\{\beta^{(\varepsilon)}J\}} = \sum_{\{\sigma_i=\pm 1\}} \sigma_k\sigma_l \exp(-\beta H_{\mathrm{I}}^{(0)}) \Big/ \sum_{\{\sigma_i=\pm 1\}} \exp(-\beta H_{\mathrm{I}}^{(0)}) \tag{2.9}$$

and

$$\beta H_{\mathrm{I}}^{(0)} = -\sum_{(ij)} \beta_{ij}^{(\varepsilon)} J_{ij}\sigma_i\sigma_j. \tag{2.10}$$

Invoking theorem 1 given by Horiguchi and Morita (1979), we have the following inequality

$$|\langle\langle s_k s_l\rangle_{N,0}^{\{\beta J\}}\rangle_c| < \langle s_k s_l\rangle_{N,0}^{\{\beta J^{\mathrm{M}}\}} \lim_{\varepsilon\to +0} \langle\langle\sigma_k\sigma_l\rangle_{N,0}^{\{|\beta^{(\varepsilon)}J|\}}\rangle_c \tag{2.11}$$

where the superscripts $\{\beta J^{\mathrm{M}}\}$ and $\{|\beta^{(\varepsilon)}J|\}$ denote the sets $\{\beta J_{ij}^{\mathrm{M}}\}$ and $\{|\beta_{ij}^{(\varepsilon)}J_{ij}|\}$, respectively, and J_{ij}^{M} is the maximum value of $|J_{ij}|$ for $J_{ij} \in \mathcal{J}_{ij} \cup \mathcal{J}_{ij}^*$. When J_{ij}^{M} does not exist, we have S^2 in place of $\langle s_k s_l\rangle_{N,0}^{\{\beta J^{\mathrm{M}}\}}$ in (2.11). It was shown by Fisher (1967) that the correlation function $\langle\sigma_k\sigma_l\rangle_{N,0}^{\{\gamma\}}$ for the set $\{\gamma_{ij}\}$ of the fixed values $\gamma_{ij} > 0$ is bounded above by the generating function $C_{kl}(\{\gamma_{ij} > 0\}, \Lambda)$ for the self-avoiding walks from site k to l on Λ, which is expressed as follows (Morita 1979)

$$C_{kl}(\{\gamma_{ij} > 0\}, \Lambda) = \sum_{\Gamma(k,l)} \prod_{(ij) \text{ on } \Gamma(k,l)} \tanh \gamma_{ij} \tag{2.12}$$

where the summation runs over all self-avoiding walks from site k to l, and the product is taken over all the bonds (i, j) on the route of the walk $\Gamma(k, l)$. Then we have

$$|\langle\langle s_k s_l\rangle_{N,0}^{\{\beta J\}}\rangle_c| \leq \langle s_k s_l\rangle_{N,0}^{\{\beta J^{\mathrm{M}}\}} \sum_{\Gamma(k,l)} \prod_{(ij) \text{ on } \Gamma(k,l)} \langle w_{ij}\rangle_c \tag{2.13}$$

where

$$\langle w_{ij}\rangle_c = \lim_{\varepsilon\to+0} \langle \tanh|\beta_{ij}^{(\varepsilon)} J_{ij}|\rangle_c$$
$$= \left\langle \left| \frac{\tilde{P}_{ij}(J_{ij}) - \tilde{P}_{ij}(-J_{ij})}{\tilde{P}_{ij}(J_{ij}) + \tilde{P}_{ij}(-J_{ij})} \right| \right\rangle_c . \tag{2.14}$$

We now restrict ourselves to the system with only the nearest-neighbour exchange integrals whose probability distribution $\tilde{P}_{ij}(J_{ij})$ are equal to the same distribution $\tilde{P}(J_{ij})$ for all the nearest neighbours (i, j). Then we have in the limit as $N\to\infty$

$$\lim_{N\to\infty} |\langle\langle s_k s_l\rangle_{N,0}^{\{\beta J\}}\rangle_c| \leqslant \langle s_k s_l\rangle^{\{\beta JM\}} f(k, l; \langle w\rangle_c) \tag{2.15}$$

where

$$\langle s_k s_l\rangle^{\{\beta JM\}} = \lim_{N\to\infty} \langle s_k s_l\rangle_{N,0}^{\{\beta JM\}} \tag{2.16}$$

$$\langle w\rangle_c = \left\langle \left| \frac{\tilde{P}(J_{ij}) - \tilde{P}(-J_{ij})}{\tilde{P}(J_{ij}) + \tilde{P}(-J_{ij})} \right| \right\rangle_c \tag{2.17}$$

and

$$f(k, l; \langle w\rangle_c) = \sum_{n=1}^{\infty} Q_n(k, l) \langle w\rangle_c^n . \tag{2.18}$$

$Q_n(k, l)$ is the total number of the self-avoiding walks of n steps from site k to l (Fisher 1967, Morita 1979). For the zero-field susceptibility defined by

$$\chi(0) = \lim_{N\to\infty} \frac{\beta}{N} \sum_k \sum_l \langle\langle s_k s_l\rangle_{N,0}^{\{\beta J\}}\rangle_c \tag{2.19}$$

we have

$$\chi(0) \leqslant \lim_{N\to\infty} \lim_{\varepsilon\to+0} \frac{\beta}{N} S^2 \sum_k \sum_l \langle\langle \sigma_k \sigma_l\rangle_{N,0}^{\{|\beta^{(\varepsilon)} J|\}}\rangle_c \tag{2.20}$$

$$\leqslant \beta S^2 g(\langle w\rangle_c) \tag{2.21}$$

where

$$g(\langle w\rangle_c) = 1 + \sum_{n=1}^{\infty} c_n \langle w\rangle_c^n . \tag{2.22}$$

Here c_n is the total number of self-avoiding walks of n steps, starting from a site far from the surface (Fisher 1967, Morita 1979).

We assumed that the phase transitions between the paramagnetic phase and the ordered phases and between the spin-glass phase and the ordered phases are second-order transitions and that the zero-field susceptibility diverges at these transition points from the above. Here we designate the spin-glass phase the one with the zero long-range order parameter and the non-zero Edwards–Anderson order parameter (Edwards and Anderson 1975). Under this postulation, we have a sufficient condition of $\tilde{P}(J_{ij})$ for disappearance of the ordered states as follows

$$\left\langle \left| \frac{\tilde{P}(J_{ij}) - \tilde{P}(-J_{ij})}{\tilde{P}(J_{ij}) + \tilde{P}(-J_{ij})} \right| \right\rangle_c \leqslant \tanh(J/k_B \tilde{T}_C) \tag{2.23}$$

where $\tilde{T}_C$ is the Curie temperature calculated by the self-avoiding walk approximation for the Ising model of spin ± 1 with the nearest-neighbour exchange integral $J>0$.

The same discussions are also applied to the system of the random-bond n-vector model under the zero external field on a finite set Λ of N lattice sites. The Hamiltonian of the system is given by (Stanley 1974)

$$H_n^{(0)} = -\sum_{(ij)} J_{ij}\, s_i \cdot s_j \tag{2.24}$$

where s_i is the n-dimensional classical spin of unit magnitude for the site i:

$$s_i = (s_i^{(1)}, s_i^{(2)}, \ldots, s_i^{(n)}) \qquad |s_i| = 1. \tag{2.25}$$

J_{ij} are the exchange integrals for the pair of sites i and j and are mutually independent, quenched random variables whose probability distributions are denoted by $\tilde{P}_{ij}(J_{ij})$. For the configurational average of the correlation function of spins s_k^{α} and s_l^{β} for $k \neq l$ defined by

$$\langle\langle s_k^{\alpha} s_l^{\beta}\rangle_{N,0}^{\{\beta J\}}\rangle_c = \langle \operatorname{Tr} s_k^{\alpha} s_l^{\beta} \exp(-\beta H_n^{(0)})/\operatorname{Tr} \exp(-\beta H_n^{(0)})\rangle_c \tag{2.26}$$

we have instead of (2.13)

$$|\langle\langle s_k^{\alpha} s_l^{\beta}\rangle_{N,0}^{\{\beta J\}}\rangle| \leqslant \sum_{\Gamma(k,l)} \prod_{(ij) \text{ on } \Gamma(k,l)} \langle w_{ij}\rangle_c. \tag{2.27}$$

Restricting ourselves to the system with only nearest-neighbour exchange integrals, whose probability distribution is denoted by $\tilde{P}(J_{ij})$, we have

$$\lim_{N\to\infty} |\langle\langle s_k^{\alpha} s_l^{\beta}\rangle_{N,0}^{\{\beta J\}}\rangle| \leqslant f(k, l; \langle w\rangle_c). \tag{2.28}$$

For the zero-field susceptibility defined by

$$\chi^{\alpha\beta}(0) = \lim_{N\to\infty} \frac{\beta}{N} \sum_k \sum_l \langle\langle s_k^{\alpha} s_l^{\beta}\rangle_{N,0}^{\{\beta J\}}\rangle_c \tag{2.29}$$

we have

$$\chi^{\alpha\beta}(0) \leqslant \beta g(\langle w\rangle_c). \tag{2.30}$$

Here $\langle w\rangle_c$, $f(k, l; \langle w\rangle_c)$ and $g(\langle w\rangle_c)$ are given by (2.17), (2.18) and (2.22), respectively. The same condition (2.23) is obtained for disappearance of the ordered states under the above postulation.

3. Upper bound to the spontaneous magnetisation

We consider the system of the random-bond Ising model of general spin S under the non-zero external field. The Hamiltonian is given by

$$H_S = H_S^{(0)} - h \sum_i \mu_i s_i. \tag{3.1}$$

$H_S^{(0)}$ is given by (2.1), h is the external field and μ_j is the magnetic moment of the spin on the site j. We define the thermodynamic limit for the absolute value of the configurational average of the canonical average of the spin s_k on the site k as follows

$$|\langle\langle s_k\rangle^{\{\beta J\},\{\mu\}}\rangle_c| = \lim_{h\to+0} \lim_{N\to\infty} |\langle\langle s_k\rangle_{N,h,B_0}^{\{\beta J\},\{\mu\}}\rangle_c| \tag{3.2}$$

where

$$\langle s_k\rangle_{N,h,B_0}^{\{\beta J\},\{\mu\}} = \mathrm{Tr}\, s_k\, \mathrm{e}^{-\beta H_S}/\mathrm{Tr}\, \mathrm{e}^{-\beta H_S}. \tag{3.3}$$

Superscripts $\{\mu\}$ represent the set $\{\mu_i\}$ of μ_i, and B_0 denotes the boundary condition that the boundary spins are not coupled with the outer system even if it exists. We consider a subset Λ_1 of Λ and denote the number of the total sites of Λ_1 by N_1. It is assumed that Λ_1 contains the site i and N_1 is so large that we make N_1 tend to infinity after making N infinity.

We consider an auxiliary Hamiltonian

$$H_S^{(a)} = H_S^{(0)} - h \sum_{i\in\Lambda\backslash\Lambda_1} \mu_i s_i \tag{3.4}$$

and the canonical average of spin s_k in the system described by $H_S^{(a)}$:

$$\langle s_k\rangle_{N,(h,0),B_0}^{\{\beta J\},\{\mu\}} = \mathrm{Tr}\, s_k \exp(-\beta H_S^{(a)})/\mathrm{Tr}\exp(-\beta H_S^{(a)}). \tag{3.5}$$

We could show in paper I that

$$\lim_{h\to+0}\lim_{N\to\infty} \langle\langle s_k\rangle_{N,h,B_0}^{\{\beta J\},\{\mu\}}\rangle_c = \lim_{h\to+0}\lim_{N\to\infty} \langle\langle s_k\rangle_{N,(h,0),B_0}^{\{\beta J\},\{\mu\}}\rangle_c. \tag{3.6}$$

Using $\alpha_{ij}^{(\varepsilon)}$ and $\beta_{ij}^{(\varepsilon)}$ defined by (2.6) and (2.7) instead of α_{ij} and β_{ij} in paper II, we show the following inequality by the same procedure as taken in paper II

$$|\langle\langle s_k\rangle_{N,(h,0),B_0}^{\{\beta J\},\{\mu\}}\rangle_c| \leqslant \lim_{\varepsilon\to+0} \langle|\langle\sigma_k\rangle_{N_1,\tilde{h},B_1}^{\{\beta^{(\varepsilon)}J\},(\tilde{\mu})}|\rangle_c \langle s_k\rangle_{N,(h,0),B_0}^{\{\beta J^{\mathrm{M}}\},\{|\mu|\}} \tag{3.7}$$

where

$$\langle\sigma_k\rangle_{N_1,\tilde{h},B_1}^{\{\beta^{(\varepsilon)}J\},(\tilde{\mu})} = \sum_{\{\sigma_i=\pm1\}} \mathrm{e}^{-\beta H_1}\sigma_k \Big/ \sum_{\{\sigma_i=\pm1\}} \mathrm{e}^{-\beta H_1} \tag{3.8}$$

and

$$\beta H_1 = -\sum_{\substack{(ij)\\ i,j\in\Lambda_1}} \beta_{ij}^{(\varepsilon)} J_{ij}\sigma_i\sigma_j - \tilde{\beta}\tilde{h}\tilde{\mu}\sum_{\substack{i\\ i\in\Lambda_1}}\sigma_i - \sum_{\substack{(ij)\\ i\in\Lambda_1, j\in\Lambda\backslash\Lambda_1}} \beta_{ij}^{(\varepsilon)} J_{ij}\sigma_i. \tag{3.9}$$

Here $\tilde{\beta}$, $\tilde{h}$ and $\tilde{\mu}$ are positive. Superscript $\{|\mu|\}$ represents the set $\{|\mu_i|\}$ of $|\mu_i|$. B_1 expresses the boundary condition that the spins which belong to $\Lambda\backslash\Lambda_1$ and interact with a spin σ_i for $i\in\Lambda_1$ are all plus one.

By applying theorem 1 of Horiguchi and Morita (1979) to the first factor on the right-hand side of (3.7), we have

$$|\langle\langle s_k\rangle_{N,(h,0),B_0}^{\{\beta J\},\{\mu\}}\rangle_c| \leqslant \lim_{\varepsilon\to+0} \langle\langle\sigma_k\rangle_{N_1,\tilde{h},B_1}^{\{|\beta^{(\varepsilon)}J|\},(\tilde{\mu})}\rangle_c \langle s_k\rangle_{N,(h,0),B_0}^{\{\beta J^{\mathrm{M}}\},\{|\mu|\}}. \tag{3.10}$$

Taking the limit as $N\to\infty$ first and then $h\to+0$ on both sides of equation (3.10), we have

$$|\langle\langle s_k\rangle^{\{\beta J\},\{\mu\}}\rangle_c| \leqslant \lim_{\varepsilon\to+0} \langle\langle\sigma_k\rangle_{N_1,\tilde{h},B_1}^{\{|\beta^{(\varepsilon)}J|\},(\tilde{\mu})}\rangle_c \langle s_k\rangle^{\{\beta J^{\mathrm{M}}\},\{|\mu|\}} \tag{3.11}$$

for any N_1 and $\tilde{h}>0$, where we took equation (3.6) into account. According to a theorem by Lebowitz and Martin-Löf (1972), we have

$$\lim_{\tilde{h}\to+0}\lim_{N_1\to\infty} \langle\sigma_k\rangle_{N_1,\tilde{h},B_1}^{\{\gamma\},(\tilde{\mu})} = \lim_{\tilde{h}\to+0}\lim_{N_1\to\infty} \langle\sigma_k\rangle_{N_1,\tilde{h},B_0}^{\{\gamma\},(\tilde{\mu})} \tag{3.12}$$

for the set $\{\gamma_{ij}\}$ of the fixed values $\gamma_{ij} = |\beta_{ij}^{(\varepsilon)}J_{ij}|$ with an arbitrary ε. Thus we have

$$\lim_{\tilde{h}\to+0}\lim_{N_1\to\infty}\lim_{\varepsilon\to+0}\langle\langle\sigma_k\rangle_{N_1,\tilde{h},B_1}^{\{|\beta^{(\varepsilon)}J|\},(\tilde{\mu})}\rangle_c = \lim_{\tilde{h}\to+0}\lim_{N_1\to\infty}\lim_{\varepsilon\to+0}\langle\langle\sigma_k\rangle_{N_1,\tilde{h},B_0}^{\{|\beta^{(\varepsilon)}J|\},(\tilde{\mu})}\rangle_c. \tag{3.13}$$

We now introduce a ghost spin σ_0 on a ghost site 0 in order to treat the external field as a part of the exchange integrals. We consider the Hamiltonian on $\Lambda' = \Lambda \cup \{0\}$

$$\beta H_1' = -\sum_{(ij)} |\beta_{ij}^{(\varepsilon)}J_{ij}|\sigma_i\sigma_j - \sum_i |\beta_{i0}^{(\varepsilon)}J_{i0}|\sigma_i\sigma_0 \tag{3.14}$$

where $|\beta_{i0}^{(\varepsilon)}J_{i0}|$ is chosen to be equal to $\tilde{\beta}\tilde{h}\tilde{\mu} > 0$. We have

$$\langle\sigma_k\rangle_{N,\tilde{h},B_0}^{\{|\beta^{(\varepsilon)}J|\},(\tilde{\mu})} = \langle\sigma_k\sigma_0\rangle_{N+1,0,B_0}^{\{|\beta^{(\varepsilon)}J|,|\beta_{i0}^{(\varepsilon)}J_{i0}|\}} \tag{3.15}$$

where the right-hand side is the average of $\sigma_k\sigma_0$ for the system described by the Hamiltonian (3.14). On the right-hand side of (3.15), $\{|\beta^{(\varepsilon)}J|, |\beta_{i0}^{(\varepsilon)}J_{i0}|\}$ represents the union of the sets $\{|\beta_{ij}^{(\varepsilon)}J_{ij}|\}$ and $\{|\beta_{i0}^{(\varepsilon)}J_{i0}|\}$. We estimate the right-hand side in terms of the self-avoiding walk approximation. It we denote by l the site just before site 0 on the route from site k to 0, we have

$$\langle\sigma_k\rangle_{N,\tilde{h},B_0}^{\{|\beta^{(\varepsilon)}J|\},(\tilde{\mu})} \leqslant \tanh(\tilde{\beta}\tilde{h}\tilde{\mu}) \sum_{l\neq 0} C_{kl}(\{|\beta_{ij}^{(\varepsilon)}J_{ij}|\}; \Lambda). \tag{3.16}$$

When we restrict ourselves to the system with only the nearest-neighbour exchange integrals, we have

$$\lim_{\tilde{h}\to+0}\lim_{N\to\infty}\lim_{\varepsilon\to+0}\langle\langle\sigma_k\rangle_{N,\tilde{h},B_0}^{\{|\beta^{(\varepsilon)}J|\},(\tilde{\mu})}\rangle_c \leqslant \lim_{\tilde{h}\to+0}\tanh(\tilde{\beta}\tilde{h}\tilde{\mu})g(\langle w\rangle_c) \tag{3.17}$$

where $\langle w\rangle_c$ and $g(\langle w\rangle_c)$ are defined by (2.17) and (2.22), respectively. From (3.11), (3.13) and (3.17) we have

$$|\langle\langle s_k\rangle^{\{\beta J\},\{\mu\}}\rangle_c| \leqslant \langle s_k\rangle^{\{\beta J^M\},\{|\mu|\}} \lim_{\tilde{h}\to+0}\tanh(\tilde{\beta}\tilde{h}\tilde{\mu})g(\langle w\rangle_c). \tag{3.18}$$

We define the long-range order parameter in the system with only nearest-neighbour exchange integrals by suitably choosing the signs of μ_i in $\{\mu_i\}$ in (3.1) as follows:

$$m\{\mu_i\} = \lim_{h\to+0}\lim_{N\to\infty}\langle\langle s_k\rangle_{N,h,B_0}^{\{\beta J\},\{\mu\}}\rangle_c. \tag{3.19}$$

From (3.18), we see that the long-range order parameter is zero as long as $g(\langle w\rangle_c)$ is finite. In this way, we again find the sufficient condition (2.23) for disappearance of the long-range orders. This condition (2.23) is investigated in detail in the next section.

The above arguments are also applied to the system of the random-bond n-vector model

$$H_n = H_n^{(0)} - h\sum_i\sum_\alpha \mu_i^{(\alpha)}s_i^{(\alpha)} \tag{3.20}$$

where $H_n^{(0)}$ is given by (2.24). We obtain

$$|\langle\langle s_k^{(\alpha)}\rangle^{\{\beta J\},\{\mu^{(\alpha)}\}}\rangle_c| \leqslant \lim_{\tilde{h}\to+0}\tanh(\tilde{\beta}\tilde{h}\tilde{\mu})g(\langle w\rangle_c) \tag{3.21}$$

for the system with only nearest-neighbour exchange integrals. Superscript $\{\mu^{(\alpha)}\}$ represents the set $\{\mu_i^{(\alpha)}\}$ of $\mu_i^{(\alpha)}$. Defining the long-range order parameter by choosing $\{\mu_i^{(\alpha)}\}$ suitably, we also obtain the sufficient condition (2.23) for disappearance of the ordered states.

4. Disappearance of the long-range order

A sufficient condition for the disappearance of the spontaneous long-range order is obtained in § 3 in the following form

$$\tanh^{-1}\left[\left\langle\left|\frac{\tilde{P}(J_{ij})-\tilde{P}(-J_{ij})}{\tilde{P}(J_{ij})+\tilde{P}(-J_{ij})}\right|\right\rangle_{c}\right]\leq\frac{J}{k_{B}\tilde{T}_{C}} \tag{4.1}$$

for the random-bond Ising model and for the random-bond n-vector model. $\tilde{T}_C$ here is the Curie temperature calculated in the self-avoiding walk approximation for the Ising model of spin ± 1 with the nearest-neighbour exchange integral $J>0$, that is to say, $\tanh(J/k_B\tilde{T}_C)$ is equal to the inverse of the self-avoiding walk limit. Its value was given by Domb for several two- and three-dimensional lattices (Domb 1970).

We investigate the condition (4.1) for several types of the probability distribution of J_{ij}.

4.1. Discrete distribution of three delta functions

Our first example is the probability distribution expressed formally by

$$\tilde{P}(J_{ij})=p\delta(J_{ij}-J)+r\delta(J_{ij})+q\delta(J_{ij}+J) \tag{4.2}$$

where $J>0$ and $p+q+r=1$. Equation (4.1) is expressed as

$$\tanh^{-1}[|p-q|]\leq J/k_{B}\tilde{T}_{C}. \tag{4.3}$$

For $r=0$, this equation is nothing other than the one obtained previously except that $\tilde{T}_C$ there was the exact Curie temperature (Horiguchi and Morita 1981, 1982). In terms of the ratio x of the mean $\bar{J}$ and the standard deviation σ, we have

$$\tanh^{-1}\left[\frac{|x|(1-r)^{1/2}}{(1+x^{2})^{1/2}}\right]\leq\frac{J}{k\tilde{T}_{C}}. \tag{4.4}$$

4.2. Gaussian distribution

Equation (4.1) is expressed as

$$\tanh^{-1}\left[\operatorname{erf}\left(\frac{|x|}{\sqrt{2}}\right)\right]\leq\frac{J}{k_{B}\tilde{T}_{C}} \tag{4.5}$$

where $x=\bar{J}/\sigma$, $\bar{J}$ is the mean and σ is the standard deviation, $\operatorname{erf}(x)$ is the error function (Magnus *et al* 1966).

4.3. Rectangular distribution

For $\tilde{P}(J_{ij})$ given by

$$\tilde{P}(J_{ij})=\begin{cases}1/(\beta-\alpha) & \alpha<J_{ij}<\beta\\ 0 & \text{otherwise}\end{cases} \tag{4.6}$$

we have

$$\tanh^{-1}(|x|/\sqrt{3})\leq J/k_{B}\tilde{T}_{C} \tag{4.7}$$

where $x=\bar{J}/\sigma$, $\bar{J}=(\alpha+\beta)/2$ and $\sigma=(\beta-\alpha)/2\sqrt{3}$.

4.4. Quadrangular distribution

For $\tilde{P}(J_{ij})$ given by

$$\tilde{P}(J_{ij}) = \begin{cases} 1/2b + aJ_{ij} & |J_{ij}| \leqslant b \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

we have

$$\tanh^{-1}\left[\frac{\sqrt{3}}{2}\frac{|x|}{(1+x^2)^{1/2}}\right] \leqslant \frac{J}{k_B\tilde{T}_C} \tag{4.9}$$

where $x = \bar{J}/\sigma$, $\bar{J} = \frac{2}{3}ab^3$ and $\sigma = (\frac{1}{3}b^2 - \frac{4}{9}a^2b^6)^{1/2}$.

4.5. Lorentzian distribution

Equation (4.1) is expressed as

$$\tanh^{-1}[(2/\pi)\tan^{-1}|x|] \leqslant J/k_B\tilde{T}_C \tag{4.10}$$

where $x = \bar{J}/\sigma$, $\bar{J}$ is the median and σ the width.

The left-hand side of equation (4.3) with $r = 0$ and $r = 0.5$, and of equations (4.4), (4.5), (4.7), (4.9) and (4.10) are shown as functions of x in figure 1. For the case of the quadrangular distribution, there is no ordered state in the system on the hexagonal

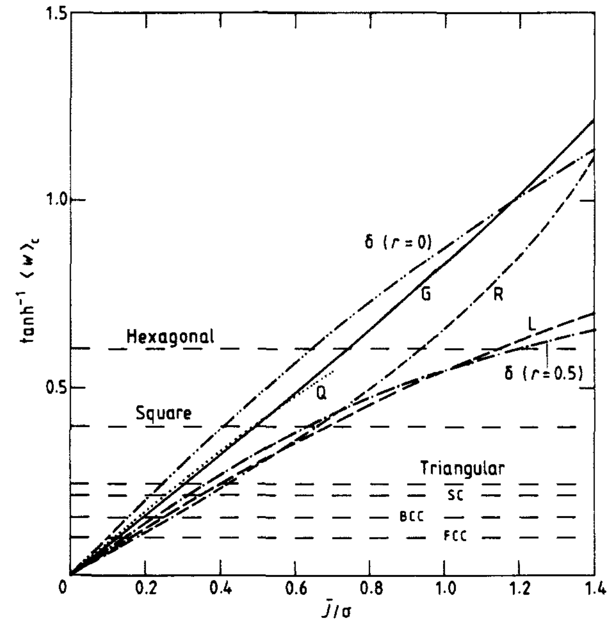

Figure 1. The graph of $\tanh^{-1}\langle w\rangle_c$ given by (2.17) for several types of probability distribution of J_{ij}. The double chain curve is for the discrete distribution of three delta functions with $r = 0$ and the chain curve for that with $r = 0.5$. The full curve is for the Gaussian distribution, the two-dash dotted curve for the rectangular distribution and the broken curve for the Lorentzian distribution. The dots are for the quadrangular distribution. $J/k_B\tilde{T}_C$ is shown by horizontal broken lines for the hexagonal, square, triangular, SC, BCC and FCC lattices.

lattice. This situation also occurs for the case of the triangular distribution discussed in paper II. The critical values of x are given in table 1.

Table 1. The lower bound to the critical value of $\bar{J}/\sigma$ for the ferromagnetic state. $\bar{J}$ is the mean and σ is the standard deviation for the discrete distribution of the three delta functions (δ, $r=0$ and δ, $r=0.5$), the Gaussian distribution (G), the quadrangular distribution (Q) and the rectangular distribution (R). $\bar{J}$ is the median and σ is the width for the Lorentzian distribution (L).

	δ, $r=0$	δ, $r=0.5$	G	Q	R	L
Hexagonal	0.6433	1.1882	0.7405	—	0.9371	1.1379
Square	0.4095	0.6349	0.4944	0.4867	0.6564	0.6773
Triangular	0.2482	0.3624	0.3066	0.2896	0.4172	0.3975
SC	0.2186	0.3168	0.2709	0.2545	0.3699	0.3486
BCC	0.1550	0.2219	0.1932	0.1797	0.2653	0.2453
FCC	0.1002	0.1423	0.1252	0.1158	0.1726	0.1578

We investigate in detail the diluted Ising model with competing interactions. $\tilde{P}(J_{ij})$ is given by (4.2). From theorem 2 in paper I, we have

$$|\langle\langle s_i\rangle^{\{\beta J\},\{\mu\}}\rangle_{\rm c}| \leqslant \langle s_i\rangle^{(\beta J),\{|\mu|\}}\langle\sigma_i\rangle^{\{(1-r)|\beta_1|J\},(\tilde{\mu})} \tag{4.11}$$

$$|\langle\langle s_i\rangle^{\{\beta J\},\{\mu\}}\rangle_{\rm c}| \leqslant \langle\langle s_i\rangle^{\{\beta|J|\},\{|\mu|\}}\rangle_{\rm c}\langle\sigma_i\rangle^{(|\beta_1|J),(\tilde{\mu})} \tag{4.12}$$

where $\beta_1 = \ln[p/(1-p-r)]/2J$. $\langle\langle s_i\rangle^{\{\beta|J|\}\{|\mu|\}}\rangle_{\rm c}$ represents the quantity $\langle\langle s_i\rangle^{\{\beta J\}\{|\mu|\}}\rangle_{\rm c}$ for the diluted ferromagnetic system in which $\tilde{P}(J_{ij})$ is non-zero for $J_{ij}=J>0$ and 0 with probability $1-r$ and r, respectively. In the preceding section, we showed that

$$|\langle\langle s_i\rangle^{\{\beta J\},\{\mu\}}\rangle_{\rm c}| \leqslant \langle s_i\rangle^{(\beta J),\{|\mu|\}} \lim_{\tilde{h}\to+0} \tanh(\tilde{\beta}\tilde{h}\tilde{\mu})g(\langle w\rangle_{\rm c}). \tag{4.13}$$

Thus, $|\langle\langle s_i\rangle^{\{\beta J\},\{\mu\}}\rangle_{\rm c}|$ is bounded above by the smallest quantity of the right-hand sides of equations (4.11)–(4.13). The right-hand side of (4.11) is zero when $|\beta_1|(1-r) \leqslant 1/k_{\rm B}T_{\rm C}$ where $T_{\rm C}$ is the exact Curie temperature of the Ising model of spin ±1 with the nearest-neighbour exchange integral $J>0$. The right-hand side of equation (4.12) is zero when $r>r_{\rm c}$, where $r_{\rm c}$ is the critical concentration of the percolation of the bond dilution (Shante and Kirkpatrick 1971). The right-hand side of equation (4.13) is zero when (4.3) is satisfied. These give lower and upper bounds to the critical concentrations of the ferromagnetic bonds for the ferromagnetic and the antiferromagnetic state, respectively. They are shown in figure 2 for the system on the hexagonal, square and SC lattices.

5. Concluding remarks

In terms of the self-avoiding walk approximation, we obtained upper bounds to the zero-field susceptibilities and spontaneous magnetisations for the systems of the random-bond Ising model of general spin S and the random-bond n-vector model. In these systems, the exchange integrals are assumed to be mutually independent, quenched random variables. Their probability distribution $\tilde{P}(J_{ij})$ is either discrete or

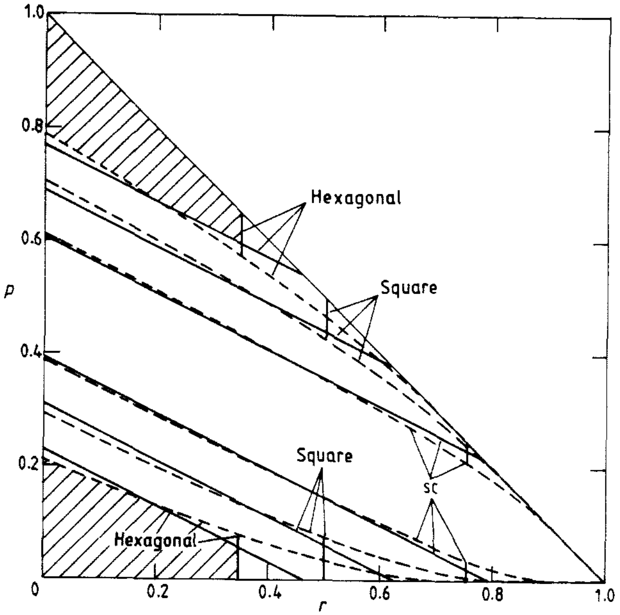

Figure 2. The graph of lower and upper bounds to the critical concentrations of the ferromagnetic bonds for the ferromagnetic and the antiferromagnetic state, respectively. The thick full vertical lines are determined by (4.11), the broken curves by (4.12) and the other thick full curves by (4.13). For example, the system on the hexagonal lattice cannot have any long-range order except in the hatched regions.

continuous. From the bounds, we found a sufficient condition for disappearance of the long-range order.

In the previous paper (Horiguchi and Morita 1982), a restriction was imposed on $\tilde{P}(J_{ij})$ such that $\tilde{P}_{ij}(J_{ij})$ is zero whenever $\tilde{P}_{ij}(-J_{ij})$ is zero. In the present paper, we could partly remove this restriction, i.e., when $\tilde{P}_{ij}(J_{ij})$ is a continuous function and the measure of the set of J_{ij}, in which $\tilde{P}_{ij}(-J_{ij})$ is zero and $\tilde{P}_{ij}(J_{ij})$ is non-zero, is finite, we could show that there is no spontaneous long-range order in the system as long as equation (4.1) is satisfied.

It is also of interest to find a sufficient condition for disappearance of the long-range order for the systems in which $\tilde{P}_{ij}(J_{ij})$ is non-zero for $J_{ij}=-J_B<0$ and for $J_{ij}=J_A>0$ with respective probabilities p and $1-p$, where $J_A \neq J_B$.

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